Numerical Models in Geomechanics

Edited by
G.N. Pande
University College of Swansea
Swansea, UK

and

W. F. Van Impe
Ghent State University
Ghent, Belgium

M. JACKSON & SON (PUBLISHERS) LTD.
REDRUTH, CORNWALL, ENGLAND
1986
Interaction between the bottom of cylindrical tank and soil

M. T. HEINISUO & K. A. MIETTINEN
Tampere University of Technology, Finland

ABSTRACT: The frictionless contact between the bottom plate of an elastic cylindrical tank and an elastic soil is considered. The contact problem is solved by reducing it to a quadratic optimization problem with some constraints. In the force method formulation of the optimization problem the constraint is that there is no tension in the contact zone. In accordance with the Kuhn-Tucker condition the contact zone is impenetrable. The contact problem is solved numerically and some limiting cases are solved analytically. The classical shell and plate theory is used in the solution for the mechanics of the tank. The mechanics of the soil is modelled as an elastic half-space or a Winkler subgrade. Numerical examples are presented. The elastic constants of the soil and the initial gap between the plate and the soil are the parameters varied in numerical examples.

1. INTRODUCTION

The interaction between structures and soil is one of the most important and difficult problems in civil engineering. The problem is important because this interaction reflects to the mechanics of the whole structure. The problem is also difficult because there are often some nonlinearities in the problem. Here the materials are supposed to be linearly elastic and the nonlinearity arises from the frictionless contact between the bottom plate of cylindrical tank and the soil.

Issa (Issa, 1985) studied recently the interaction between cylindrical tank and soil. In that paper was no non-linear problem. Girkman (Girkman, 1963) presented an approximative solution to a special case considered here. By numerical methods we can solve this kind of contact problems. The authors (Heinisuo, Miettinen, 1985) presented a short review concerning the contact problems of plates and other kinds of supports. Ascione and Grimaldi (Ascione, Grimaldi, 1984) used recently a similar numerical method to that used in this paper. A variational formulation is used in that ref.

1.1 Formulation of the problem

The frictionless contact between the bottom plate of an elastic cylindrical tank and an elastic soil (modelled as a half-space or a Winkler subgrade) is considered. The tank is simply supported at the corner joining the cylindrical shell and the bottom plate. Moreover, the plate is supported by the soil under the tank. When the tank is filled with some liquid the shell bends the plate upwards. The hydrostatic pressure of the liquid bends the plate downwards and if the initial gap between the plate and soil is small then the plate is in contact with soil only partially. If there is no initial gap or no initial preforce on the potential contact surface then the contact is linear and the contact zone is unchanged if the loading increases proportionally (Parland, 1968).

The contact problem considered can be solved by reducing it to a quadratic optimization problem with some constraints. Due to the rotational symmetry of the problem we can solve analytically the mechanics of the plate and soil separately when we use the force method described in (Heinisuo, Miettinen, 1985). Numerical calculations are needed only when solving the contact problem. In the force method formulation of the optimization problem the constraint is
that there is no tension in the contact zone. In accordance with the Kuhn-Tucker
condition, the contact zone is impenetrable. The authors used the same
method when solving a linear contact problem. In this paper the contact is not
linear because of the possible gap
between the plate and soil. One special
case of the problem is solved firstly
analytically.

2. RIGID AND UNIFORMLY SAGGED FOUNDATION

Consider the simply supported cylindrical
tank on the uniformly sagged rigid
foundation (fig. 1). We neglect the

\[ w(a) = w_0, \quad dw(a)/dr = 0, \quad M_r(a) = 0, \]
\[ w(b) = 0, \quad M_r(b) = -M, \quad X + dw(b)/dr = 0, \]

(3)

where the bending moment \( M_r \) is

\[ M_r = K \left( \frac{a^2}{2} \right) \frac{\partial^2 w}{\partial r^2} \]

(4)

The rotation \( X \) and the bending moment \( M_r \) are dependent. We suppose that
the shell is long \( (1 + w)^2 \) 220 \( b^2t^2 \), see
fig. 1) and we neglect the effect of
vertical loading for the shell. The
parameter \( b \) is defined by

\[ b = \sqrt{\frac{3(1 - w^2)}{0.2}} \]

(5)

and we suppose that the shell and the
plate are made of the same material. \( Q \) is
the shear force of the shell in the joint
and \( Q = \sigma_1 L \). The compatibility condition of
the horizontal displacement \( \Delta r \) in the
joint is

\[ \frac{2b^2 E}{Q} = \frac{2b^2 E}{Q} = \frac{b(1 + w)}{b} \]

(6)

The rotation of the shell in the joint is

\[ X = b^2 E \]

(7)

where

\[ \lambda = \frac{1}{b^2 E} \left( \frac{1 + w}{1 + w} \right) \]

(8)

Now we have six unknown variables \( A, B, C, D, M \) and a and six equations (3). We
see that equations (3) are non-linear in
unknown a. We find that equations (3) are
linear in other unknowns. We can guess
the radius a and solve the first five
linear equations (3) and use the last as
a penalty equation. We iterate till the
required accuracy is achieved. The first
five eqs. (3) are in matrix form
Girkman derived an approximative solution for the case \( \bar{q} = 0 \). Numerical data was \( E = 210 \) 000 MPa, \( t = 6 \) mm, \( h = 7 \) mm, \( L = 12 \) m, \( b = 6 \) mm, \( v = 0.3 \), \( \rho = 0.004 \) kg/cm\(^3\), \( \rho_g = 0.006 \) kg/cm\(^3\). The result was \( b = a = 191 \) mm. The method referred to gives the solution \( b = a = 191 \) mm. The contact stress is uniform 168 kPa including the peak value of 0.0808 kN/m on the line of separation. It is observed that the normal stress in the corner due to the bending of the plate is in the example mentioned above 186.2 MPa. When the initial gap increases the stresses increase.

Fig. 2 presents the radius of the contact zone as a function of the initial gap. Fig. 3 presents the proportional support reaction as a function of the initial gap. The deflection \( w \) is the maximum deflection of the plate when there is no foundation under the plate. The peak mentioned above is, however, theoretical and it is smoothened if we take into account the shear deformation of the plate and/or the elasticity or the plasticity of the soil. In the following we take into account the elasticity of the soil.

3. NUMERICAL METHOD

The unknown contact stress on the potential contact zone is discretized into \( n \) point forces. The vectors \( x_1 \) and \( x_2 \) are the displacements of nodal points due to loadings of bodies 1 and 2. The vector \( \bar{q} \) is the initial gap and the matrices \( C_1 \) and \( C_2 \) (dimensions \( n \times n \)) are the flexibility matrices of bodies 1 and 2. The frictionless contact problem of two elastic bodies can be solved by the quadratic optimization problem

\[
\min \sum_{i=1}^{n} \left[ (C_i x) \cdot (C_i x) + p_i (x_1 + x_2) \right].
\]

This force method is suitable when the rigid body motions of bodies 1 and 2 are zero. The numerical solution of the optimization problem is computed with the simple algorithm:

Step 1: Solve \( (C_1 + C_2) x = (x_1 + x_2) \)
Step 2: If \( p > 0 \) then stop
Step 3: Set \( \min p_i = 0 \), \( i = 1, \ldots, n \)
Step 4: Go to step 1.

Next we calculate the matrices \( C_1 \) and \( C_2 \) and the vectors \( x_1 \) and \( x_2 \) for the case.

3.1 Plate and shell

The vector \( p \) includes the resultant of the uniform line load. The node \( i \) \((i=1, \ldots, n)\) is located on the ring \( r_i = (i-1/2)b/n \). The elements of the flexibility matrix \( C_1 \) are the deflections in lines \( r_1 \) when the plate is loaded by the uniform ring load \( P/r_i \). In fact we need the deflections on lines \( r_1 \) where

\[
\begin{align*}
p \cdot r_i &= \frac{P}{r_i} \quad (i=1, \ldots, n) \\
&= \frac{P}{b/n} \quad (i=1, \ldots, n).
\end{align*}
\]
\[ r_1^2 < b \text{ and can use Betti's reciprocal theorem.} \]

\[ \begin{align*}
\text{Figure 4.} & \quad C_{11}\text{j and } V_{11}. \\
\text{This solution is derived by superposition. The deflections of the} & \\
\text{simple supported plate loaded with the} & \\
\text{ring loading } P & \text{and moment } M_1 \text{ (fig. 4) can} & \text{be found in many text books e.g.} & \\
\text{(Girkan, 1963). The results are} & \\
C_{111} & = \frac{P}{8K} \left[ \frac{2r_1^2}{1+\nu} \left( \frac{1}{1+\nu} \right) \frac{r_1^2}{1-\nu} \right] - 2(r_j^2 + r_j^2) \frac{1}{b_x} \\
& + 2(1+\nu) \frac{1}{b_x} \left( \frac{1}{1+\nu} \right) \frac{r_1^2}{b_x^2}, \quad r_j^2 < b_x \text{.} \\
& \quad (13)
\end{align*} \]

\[ \text{The compatibility condition of the} \]

\[ \frac{1}{K} \frac{M_1}{M_1} \left( \frac{r_1^2}{b_x^2} \right) \frac{1}{b_x} \left( \frac{1}{1+\nu} \right) \frac{r_1^2}{b_x^2} \]

\[ \quad \Rightarrow r_j^2 > (2a) \frac{b_x^2}{b_2} \text{.} \]

\[ \quad (14) \]

\[ \text{where } a_2 = 2a_2 \left( \frac{1+\nu}{b_x} \right) \text{. In the numerical} \]

\[ \text{procedure we need the deflections of the} \]

\[ \text{plate without the foundation, the vector } \chi_1 \text{. These deflections can be calculated} \]

\[ \text{as above. The results are} \]

\[ \psi = \frac{q}{8K} \frac{2}{1+\nu} \left( \frac{1}{1+\nu} \right) \frac{r_1^2}{b_x^2} \]

\[ \quad \Rightarrow \psi_2 = \frac{1}{b_x^2} \left( \frac{1}{1+\nu} \right) \frac{r_1^2}{b_x^2} \]

\[ \quad (15) \]

\[ \text{where } \psi_2 = \psi_2 \left( \frac{q-a_2 q}{1+\nu} \right)/4/(2a_2) \text{ and} \]

\[ a_2 = \frac{1}{1+\nu}/2. \quad \text{Now we can solve the same} \]

\[ \text{problem as before numerically. The} \]

\[ \text{solution of the contact problem of the} \]

\[ \text{elastic body (the tank) and the rigid} \]

\[ \text{foundation can be approximated with the} \]

\[ \text{optimization problem by putting } C_p = 0 \text{ and} \]

\[ \chi_2 = 0. \text{ The results of such a study are} \]

\[ \text{presented in fig. 5 with various} \]

\[ \text{dimensions } n. \text{ We see that the numerical} \]

\[ \text{method gives good results for design} \]

\[ \text{purposes when } n=10 \text{ with this geometrical} \]

\[ \quad \begin{align*}
\text{Figure 5. Comparison between numerical} & \quad \text{and exact results.} \\
\text{and material data. The peak in the} & \\
\text{support reaction is difficult to handle} & \\
\text{with good accuracy but e.g. the} & \\
\text{contactzone is almost correct when } n=10. & \\
\text{Hence this peak is theoretical as} & \\
\text{mentioned above. It is also found that} & \\
\text{the contact stress in the middle point of} & \\
\text{the plate has an unrealistic "hole" when} & \\
\text{the parameter } n \text{ increases.} & \\
\text{3.2 Elastic foundation} & \\
\text{The foundation is firstly modelled to} & \\
\text{the elastic half-space with material} & \\
\text{constants } E_r \text{ and } v_r. \text{ The vector } \psi \text{ is} & \\
\text{zero. To calculate the elements of the} & \\
\text{matrix } C_p \text{ we replace the uniform line} & \\
\text{loading } P \left( \frac{1}{2(b+y)} \right) \text{ with the uniform} & \\
\text{surface loading } P \left( \frac{1}{2(b+y)} \right) \text{ on the} & \\
\text{narrow ring of width } b \text{ on the foundation.} & \\
\text{The element } C_{11} \text{ is defined} & \\
\text{as } C_{11} = \frac{1}{1+\nu} \frac{1}{b_x} \\
\text{and } C_{22} \text{ as } C_{22} = \frac{1}{1+\nu} \frac{1}{b_x} \\
\text{by} & \\
\text{where } \psi_2 \text{ is the deflection of the} & \\
\text{foundation due to the uniform ring} & \\
\text{loading. The deflection } \psi_2 \text{ can be} & \\
\text{calculated analytically and the} & \\
\text{integration is done numerically with} & \\
\text{Simpson's rule. The result is} & \\
\text{C}_{21} = \frac{1}{2 \pi} \int \frac{1}{r_3} \text{d} r_3 \text{d} r_3, \quad (16)
\end{align*} \]

\[ \text{where } \psi_2 \text{ is the deflection of the} \]

\[ \text{foundation due to the uniform ring} \]

\[ \text{loading. The deflection } \psi_2 \text{ can be} \]

\[ \text{calculated analytically and the} \]

\[ \text{integration is done numerically with} \]

\[ \text{Simpson's rule. The result is} \]

\[ \text{C}_{21} = \frac{1}{2 \pi} \int \frac{1}{r_3} \text{d} r_3 \text{d} r_3, \quad (16) \]

\[ \text{where } r_3 = (j-1/2) b/n, \quad r_3 = (j-1) b/n, \quad r_3 = 3 b/n. \]

\[ 462 \]
A similar definition holds for the elements $C_{ij}$ namely

$$C_{21} = \frac{1}{6r_i} \int (r_i \phi_p(r_i) + 4r_i \phi_p(r_i) + r_i \phi_p(r_i^2))$$

and the matrix $C_2$ is symmetric. The deflections of the uniform ring loading can be calculated by subtracting two uniform loadings acting on the circular areas with radii $r_0 = r_j$ and $r_0 = r_j$. The equations needed are

$$w_r(r) = \frac{1}{2 \pi \rho \sigma f} F_1(k), \quad r > r_0$$

$$w_r(r) = \frac{1}{2 \pi \rho \sigma f} (F_1(k) - (1 - k^2)F_2(k)), \quad r < r_0$$

where $F_1$ and $F_2$ are the complete elliptic integrals of the first and second kind. The elliptic integrals were calculated by using hypergeometric series accordant the tables of (Gradisheteyn, Ryshik, 1960). The series were calculated by using 20+INT(130k') terms, where $k'$ is the modulus.

If the foundation is modelled by the Winkler subgrade then the matrix $C_2$ is diagonal and of form

$$C_{ij} = \frac{N^2}{2 \pi \rho \sigma f (1 - \frac{1}{2})},$$

where $N$ is the modulus of the subgrade.

When the result (the vector $\phi$) is calculated, then the contact stress $\sigma$ is approximated in the nodal points and the deflections of the plate $w$ and the deflections of the plate $w$ and the foundations $w_r$ are calculated by equations

$$w_{ij} = \frac{N^2}{2 \pi \rho \sigma f (1 - \frac{1}{2})},$$

$$w = \frac{N^2}{2 \pi \rho \sigma f (1 - \frac{1}{2})},$$

Figure 6 presents the results of the study with various initial uniform gap. Figure 7 presents the results when the modulus $c$ is as a variable. It is seen that the maximum contact stress occurs in the middle of the plate when approximately $c \leq 10$ MN/m² with this numerical data. When $c > 10$ MN/m² the maximum contact stress occurs near the line of separation. It is also seen that if $c \geq 1000$ MN/m² then the result is almost the same than with the rigid foundation.

Table 1 presents the modulus of Winkler subgrade and the elastic modulus of half-space which give the same deflections $w_r(0)$ in one case.

| Table 1. | $w_r = 10$ mm, $E=210000$ MPa, $v=0.3$, $t=20$ mm, $h=20$ mm, $L_1=4000$ mm, $b=2000$ mm, $\rho_1=1400$ kg/m², $\rho_2=1800$ kg/m³, $\nu=0.2$. |
|---|---|---|---|---|
| $w_r(0)$ | 1.78 | 7.80 | 1.32 | 0.05 |
| $c$ | 5 | 20 | 50 | 1000 |
| $N$ | 9.35 | 15.8 | 112 | 2265 |

463
4. CONCLUSIONS

The proposed method seems suitable when analyzing the interaction between elastic bodies. If the rigid body motion of the bodies is known (or some true contact points can be guessed beforehand) then the force method seems suitable for the study, especially when the flexibility matrices can be calculated analytically. In this way we can solve the mechanics of the contacting bodies separately. If the analytical solution is not available then we can use the standard FEM programs. After that the contact problem can be solved with the microcomputer if the dimension of the problem is small, say n ≤ 50 although the flexibility matrices are full. This accuracy is often enough for design purposes. If the rigid body motion is not known then we can use the displacement method or other formulations. The initial gap is given in the vector and we may give it pointwise. The method proposed can be used when considering the model problem for the foundation: Winkler subgrade or half-space? The modulus of the Winkler subgrade is easy to change, because the corresponding matrix is diagonal. Finally no extra variables due to slack-variables or gap-elements are needed, which keeps the dimension of the problem small.

In further studies the elastic-plastic behavior of the foundation must be considered and some methods concerning this effect are presented in the literature. The proposed model might be the first step when analyzing the structures on the Winkler subgrade, where the modulus is changing according to the contact stresses. Also the influence of large displacements of the plate is open in many cases. This is important in cases where the initial gap is large compared with the thickness of the plate. Also the reason for the hole in contact stresses mentioned above must be studied. There are very little literature concerning the contact problems between structures and foundations in the dynamic loading.

5. ACKNOWLEDGEMENT

The financial support of Finnish Academy is gratefully acknowledged.

6. REFERENCES

Asgione L., Grimaldi A., Unilateral contact between a plate and an elastic foundation, Meccanica 19, 1984, pp. 223.